Almost Gorenstein determinantal rings of symmetric matrices

Naoki Endo

School of Political Science and Economics, Meiji University

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Ela Celikbas, Jai Laxmi, and Jerzy Weyman

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Introduction

Determinantal rings

- $m, n \geq 2$ integers
- $X = [X_{ij}]$ an $m \times n$ matrix of indeterminates over an infinite field k
- $S = k[X] = k[X_{ij} \mid 1 \leq i \leq m,\ 1 \leq j \leq n]$
- $\mathrm{I}_t(X)$ the ideal of S generated by $t \times t$-minors of X, where $2 \leq t \leq \min\{m, n\}$
- $R = S/\mathrm{I}_t(X)$

- R is CM with $\dim R = mn - (m - (t-1))(n - (t-1))$ ([Hochster-Eagon, 1970])
- R is Gorenstein $\iff$ $m = n$ ([Svanes, 1974])

Theorem 1 (Taniguchi, 2018)

Let $\mathfrak{m} = R_+$ be the graded maximal ideal of R. Then TFAE.

(1) *R is an almost Gorenstein graded (AGG) ring.*

(2) *$R_{\mathfrak{m}}$ is an almost Gorenstein local (AGL) ring.*

(3) *Either $m = n$, or $m \neq n$ and $2 = t = \min\{m, n\}$.*

- R is AGG $\implies$ $R_\mathfrak{m}$ is AGL, and the converse is NOT true in general.
- The converse holds when $R = S/\mathrm{I}_t(X)$ or $R = k[H]$ ([E-Matsuoka, 2024]).

Question 2 (Goto)

Under what conditions are the determinantal rings of symmetric matrices AG?

Determinantal rings of symmetric matrices

- $n \geq 2$ integer
- $X = [X_{ij}]$ an $n \times n$ symmetric matrix of indeterminates over an infinite field k
- $S = k[X] = k[X_{ij} \mid 1 \leq i, j \leq n]$
- $\mathrm{I}_{t+1}(X)$ the ideal of S generated by $(t+1) \times (t+1)$-minors of X, where $1 \leq t \leq n$
- $R = S/\mathrm{I}_{t+1}(X)$

- R is CM with $\dim R = nt - \dfrac{1}{2}t(t-1)$ ([Kutz, 1974])
- R is Gorenstein $\iff$ $n - t$ is odd ([Goto, 1979])

Main Theorem

Main Theorem (Celikbas-E-Laxmi-Weyman, 2022)

Let $\mathfrak{m} = R_+$ be the graded maximal ideal of R. Then TFAE.

(1) *R is an AGG ring.*

(2) *$R_\mathfrak{m}$ is an AGL ring.*

(3) *Either $n - t$ is odd, or $n = 3$ and $t = 1$.*

Consider

- V an $n \times t$ matrix of indeterminates over k with $\mathrm{ch}\, k = 0$
- $A = k[V]$
- $G = O(t, k)$ the orthogonal group.

Assume that G acts on A as k-automorphisms by taking V onto VH^{-1} for $H \in G$.

Then A^G is generated by the entries of the $n \times n$ symmetric matrix $Y = VV^T$ and the ideal of relations on Y is generated by the $(t+1) \times (t+1)$ minors of Y.

Corollary 3

A^G *is AGG* $\iff$ $(A^G)_\mathfrak{m}$ *is AGL* $\iff$ *either $n - t$ is odd, or $n = 3$ and $t = 1$.*

Proof of Main Theorem (2) ⇒ (3)

May assume $\operatorname{ch} k = 0$. Set $A = R_{\mathfrak{m}}$ and $\mathfrak{n} = \mathfrak{m}A$. Choose an exact sequence

$$0 \to A \xrightarrow{\varphi} \mathrm{K}_A \to C \to 0$$

s.t. $\mu_A(C) = \mathrm{e}^0_{\mathfrak{n}}(C)$. May assume $n - t$ is even, i.e., A is not Gorenstein. Then, because $\varphi(1) \notin \mathfrak{n}\,\mathrm{K}_A$, we get $\mu_A(C) = r(A) - 1$ and

$$0 \to \mathfrak{n}\varphi(1) \to \mathfrak{n}\,\mathrm{K}_A \to \mathfrak{n}C \to 0.$$

By setting $d = \dim A$, we have

$$\mu_A(\mathfrak{n}\,\mathrm{K}_A) \leq \mu_A(\mathfrak{n}) + \mu_A(\mathfrak{n}C) \leq \frac{n(n+1)}{2} + (d-1)(r(A)-1)$$

where the second follows from $\mathfrak{n}C = (f_1, f_2, \ldots, f_{d-1})C$ for $\exists\, f_i \in \mathfrak{n}$.

By taking the S-dual of a graded minimal S-free resolution

$$0 \to F_\ell \to F_{\ell-1} \to \cdots \to F_0 \to R \to 0,$$

we get the presentation of K_R. Hence

$$\mu_R(\mathfrak{m}) \cdot \operatorname{rank} F_\ell - \operatorname{rank} F_{\ell-1} \leq \mu_R(\mathfrak{m}K_R) = \mu_A(\mathfrak{n}\,\mathrm{K}_A).$$

Let X^s be the space of $n \times n$ symmetric matrices over k. Then

$$k[X^s] \cong k[\phi_{i,j} \mid 1 \le i \le j \le n]$$

where $\phi_{i,j}$ denotes the (i,j)-th coordinate function on X^s. The subvariety

$$Y_t^s = \{\phi \in X^s \mid \text{rank } \phi \le t\}$$

can be identified with the set of symmetric matrices whose minors of size $t+1$ vanish.

Hence

$$k[Y_t^s] \cong k[\phi_{i,j} \mid 1 \le i \le j \le n]/J_{t+1}$$

where J_{t+1} denotes the ideal generated by $(t+1) \times (t+1)$ minors of $\Phi = [\phi_{i,j}]$.

Thus, in order to compute the ranks of the resolution of $R = S/I_{t+1}(X)$, it comes down to studying the resolution of $k[Y_t^s]$.

Using the Schur modules, we can compute

$$\text{rank } F_{\ell-1} = n\binom{n}{t+1} - \binom{n}{t+2} \quad \text{and} \quad \text{rank } F_\ell = \binom{n}{t}.$$

This implies $n = 3$ and $t = 1$, as desired. □

Thank you for your attention.